Ergodic Theory - Week 3

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1 Birkhoff's pointwise ergodic theorem

P1. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system, $f \in L^1(\mu)$ and $a \in \mathbb{R}$. Show that, for almost all $x \in X$, the limit

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(na) f(T^n x)$$

exists.

Let $(\mathbb{T}, \mathcal{B}(\mathbb{T}), \lambda, S)$ denote the rotation $Sx = x + a \pmod 1$ on the circle $\mathbb{T}$ with the Borel σ-algebra and the Lebesgue measure and consider the function $g(y) = e(y)$. Then, a simple calculation implies that $S^n g(y) = e(y + na)$ for all $y \in \mathbb{T}$. Consider the product system $(X \times \mathbb{T}, \mathcal{B} \times \mathcal{B}(\mathbb{T}), \mu \times \lambda, T \times S)$, which is measure-preserving (see Exercise Sheet 1). We apply the pointwise ergodic theorem for the function $f \otimes g$ defined by $(f \otimes g)(x, y) = f(x)g(y)$. We infer that for almost all $(x, y) \in X \times \mathbb{T}$, the limit

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} (T \times S)^n \left(f \otimes g\right)(x, y)$$

exists. Note that

$$(T \times S)^n (f(x)g(y)) = f(T^n x) g(S^n y) = e(y + na) f(T^n x).$$

We conclude that for almost all $(x, y) \in X \times \mathbb{T}$, the limit

$$\lim_{N \to +\infty} \frac{e(y)}{N} \sum_{n=0}^{N-1} e(na) f(T^n x) \tag{1}$$

exists. Let $A \subseteq X \times \mathbb{T}$ be the set of (x, y) for which the limit in (1) exists, so that $(\mu \times \lambda)(A) = 1$. Fubini's theorem implies that

$$\int_{\mathbb{T}} \left(\int_X 1_A(x, y) d\mu(x) \right) d\lambda(y) = 1.$$

We conclude that there exists at least one $y_0 \in \mathbb{T}$ such that $\int_X 1_A(x, y_0) d\mu(x) = 1$ (actually, almost all $y \in \mathbb{T}$ satisfy this property). For this y_0, we have $1_A(x, y_0) = 1$ for almost all $x \in X$ (otherwise the integral would not equal 1), and, thus, we infer that for almost all $x \in X$, the limit

$$\lim_{N \to +\infty} \frac{e(y_0)}{N} \sum_{n=0}^{N-1} e(na) f(T^n x)$$

exists. The conclusion follows.

P2. In this exercise, we study the ergodic theorem for non-integrable functions.

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure-preserving system. Suppose $f \geq 0$ is a measurable function such that $\int f \, d\mu = +\infty$ and define

$$f^*(x) = \liminf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

as well as the set

$$A = \{x \in X \colon f^*(x) < +\infty\}$$

(a) Show that f^* is T-invariant.
Hint: Use the identity

$$\frac{N+1}{N} \left(\frac{1}{N+1} \sum_{n=0}^{N} f(T^n x) \right) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(Tx)) + \frac{1}{N} f(x).$$

> Let $x \in X$. For any $N \in \mathbb{N}$, we have the identity
>
> $$\frac{N+1}{N} \left(\frac{1}{N+1} \sum_{n=0}^{N} f(T^n x) \right) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(Tx)) + \frac{1}{N} f(x).$$
>
> Taking limits along a subsequence N_k where the left-hand side converges to $f^*(x)$, we get that
>
> $$\lim_{N_k \to +\infty} \frac{1}{N_k} \sum_{n=0}^{N_k - 1} f(T^n(Tx)) = f^*(x).$$
>
> This implies that $f^*(x) \geq f^*(Tx)$. Doing the same for the right-hand side, we get the reverse inequality
>
> $$f^*(x) \leq f^*(Tx).$$
>
> We conclude that $f^*(x) = f^*(Tx)$ and since x was arbitrary, we deduce that f^* is T-invariant.

(b) Show that the function $f^* \cdot \mathbb{1}_A$ is constant almost everywhere on X.

> Denote $g = f^* \cdot \mathbb{1}_A$ for brevity and notice that g is a measurable function that takes values in $\mathbb{R}_{\geq 0}$. We show that g is T-invariant. We know that f^* is T-invariant, which also implies that the set A in the statement is also T-invariant. Then, for any $x \in X$, we have
>
> $$g(x) = \begin{cases} f^*(x), \ x \in A \\ 0, \ x \notin A \end{cases} = \begin{cases} f^* \circ T(x), \ Tx \in A \\ 0, \ Tx \notin A \end{cases} = \left((f^* \cdot \mathbb{1}_A) \circ T \right)(x) = g \circ T(x)$$
>
> verifying our claim. It follows by the ergodicity of T that g equals a constant for almost all point $x \in X$.

(c) Conclude that $\mu(A) = 0$ and thus

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = +\infty$$

for almost all $x \in X$.
Hint: Construct an increasing sequence of bounded functions f_m that converges to f pointwise.

Suppose that $\mu(A) > 0$. Since g is constant almost everywhere, we have that $\mu(A) = 1$ and that there exists a real constant c such that $f^*(x) = c$ for a set $B \subset A$ of measure 1. For each $m \in \mathbb{N}$, define the function

$$f_m(x) = \begin{cases} f(x), & f(x) < m \\ 0, & \text{otherwise} \end{cases}.$$

Then, f_m is an increasing sequence of non-negative measurable and bounded functions such that $f_m \to f$ pointwise. The monotone convergence theorem implies that

$$\int f_m \, d\mu \to \int f \, d\mu = +\infty.$$

Now, we define

$$f_m^*(x) = \liminf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_m(T^n x)$$

and notice that $f_m^*(x) \leq f_{m+1}^*(x) \leq f^*(x)$ for all $x \in X$, since $f_m(x) \leq f_{m+1}(x) \leq f(x)$. In addition, the pointwise ergodic theorem implies that the lim inf defining f_m^* is actually a limit for almost all $x \in X$. Due to ergodicity, we have

$$f_m^*(x) = \int f_m \, d\mu$$

for almost all $x \in X$.
The inequality $f_m^*(x) \leq f^*(x)$ yields

$$\int f_m^* \, d\mu \leq \int f^* \, d\mu = \int_B f^* \, d\mu = c,$$

since $\mu(B) = 1$. Taking limits as $m \to +\infty$ and using the monotone convergence theorem, we get $c \geq \lim_{m \to +\infty} \int f_m \, d\mu = +\infty$, a contradiction. Thus, we infer that $\mu(A) = 0$ and, thus,

$$\liminf_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = +\infty$$

for almost all $x \in X$. Equivalently,

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = +\infty$$

for almost all $x \in X$.

P3. Consider the system $(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu, T)$ where μ is the Lebesgue measure and $T(x) = qx \bmod 1$ for $q \geq 2$ a fixed integer. If $x \in \mathbb{R}$ we say that x is written in its q-ary digit expansion if

$$x = \lfloor x \rfloor + \sum_{j=1}^{\infty} \frac{a_j(x)}{q^j},$$

for $\{a_j\}_{j \in \mathbb{N}} \subseteq \{0, 1, \ldots, q-1\}$, so that for all $J \in \mathbb{N}$ there exists $j \geq J$ such that $a_j \neq q - 1$.

(a) For $x \in \mathbb{R}$, prove that $a_k(x) = i$ if and only if $T^{k-1}x \in [\frac{i}{q}, \frac{i+1}{q})$ with $i \in \{0, 1, \ldots, q-1\}$. Additionally, prove that $a_k(Tx) = a_{k+1}(x)$ for all $x \in \mathbb{R}$ and $k \geq 1$.

> To keep notation simpler in our arguments, we will apply T to any $x \in \mathbb{R}$. By this, we mean that we apply the transformation on the image of x under the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ (i.e. we apply T to $\{x\}$). Note that this is well-defined, meaning that two elements x, y with the same fractional part satisfy $Tx = Ty$.
> The main observation is that if
>
> $$x = \lfloor x \rfloor + \sum_{j=1}^{\infty} \frac{a_j(x)}{q^j},$$
>
> then
>
> $$q^k x = m_k + a_k(x) + \sum_{j=k+1}^{+\infty} \frac{a_j(x)}{q^{j-k}}$$
>
> for an integer m_k. Thus,
>
> $$T^k x = \sum_{j=k+1}^{+\infty} \frac{a_j(x)}{q^{j-k}} \pmod 1. \tag{2}$$
>
> If $a_k(x) = i$ then we have that
>
> $$T^{k-1}x = \frac{a_k(x)}{q} + \sum_{j=k+1}^{\infty} \frac{a_j(x)}{q^{j-k+1}} = \frac{i}{q} + \sum_{j=k+1}^{\infty} \frac{a_j(x)}{q^{j-k+1}} \pmod 1.$$
>
> Notice that
>
> $$0 \leq \sum_{j=k+1}^{\infty} \frac{a_j(x)}{q^{j-k+1}} < \sum_{j=2}^{\infty} (q-1)q^{-j} = \frac{1}{q},$$
>
> which implies that $T^{k-1}x \in [\frac{i}{q}, \frac{i+1}{q})$. The inequality is strict, because not all digits can be equal to $q-1$ from some point onward.
> Conversely, assume that $T^{k-1}x \in [\frac{i}{q}, \frac{i+1}{q})$. This is equivalent to
>
> $$\frac{i}{q} \leq \frac{a_k(x)}{q} + \sum_{j=k+1}^{\infty} \frac{a_j(x)}{q^{j-k+1}} < \frac{i+1}{q}.$$
>
> As $\sum_{j=k+1}^{\infty} \frac{a_j(x)}{q^{j-k+1}} \in [0, 1/q)$ by our previous calculation, we have
>
> $$\frac{i-1}{q} < \frac{a_k(x)}{q} < \frac{i+1}{q},$$
>
> which implies $a_k(x) = i$.
> The final part of the statement is obvious since if $T^{k-1}y \in [\frac{i}{q}, \frac{i+1}{q})$ we know that $a_k(y) = i$, so plugging in $y = Tx$ the same statement implies that $a_{k+1}(x) = i$.

(b) Let $c_1, \ldots, c_k$ be a collection of digits in $\{0, \ldots, q-1\}$. Show that there exists a unique $i \in \{0, \ldots, q^k - 1\}$, such that

$$\{a_1(x) = c_1, \ldots, a_k(x) = c_k\} \text{ if and only if } \{x\} \in \left[\frac{i}{q^k}, \frac{i+1}{q^k}\right).$$

Derive the equivalence

$$\{a_{n+1}(x) = c_1, \ldots, a_{n+k}(x) = c_k\} \text{ if and only if } \{T^n x\} \in \left[\frac{i}{q^k}, \frac{i+1}{q^k}\right).$$

> We will prove our claim for $i = q^{k-1}c_1 + \cdots + qc_{k-1} + c_k$. We establish the second equivalence since it implies the first one. We write x in its base-q expansion
>
> $$x = \lfloor x \rfloor + \sum_{j=1}^{+\infty} \frac{a_j(x)}{q^j}.$$
>
> and observe that
>
> $$\{q^n x\} = \frac{a_{n+1}(x)}{q} + \cdots + \frac{a_{n+k}(x)}{q^k} + \frac{1}{q^{k+1}} \left(\sum_{j>n+k} \frac{a_j(x)}{q^{j-n-k-1}} \right). \tag{3}$$
>
> Observe that the last term in the sum is strictly smaller than $\frac{1}{q^k}$. This follows by bounding all digits $a_j(x)$ by $q-1$ (this inequality is strict, since there is at least one digit not equal to $q-1$) and then computing the sum of the geometric series. Thus, $\{q^n x\} \in [\frac{i}{q^k}, \frac{i+1}{q^k})$ if and only if
>
> $$\frac{q^{k-1}a_{n+1}(x) + \cdots + a_{n+k}(x)}{q^k} = \frac{i}{q^k}$$
>
> Using the uniqueness of the base q expansion of the integer i, we derive the equalities $a_{n+1}(x) = c_1, \ldots, a_{n+k}(x) = c_k$. The conclusion follows.

(c) We say that a number x is normal in base q if for any finite pattern of digits $\{c_1, \ldots, c_k\} \in \{0, q-1\}^k$, we have

$$\lim_{N\to+\infty} \frac{|\{n \leq N \colon a_n(x) = c_1, \ldots, a_{n+k-1}(x) = c_k\}|}{N} = \frac{1}{q^k},$$

where $a_n(x)$ are the digits of x in its base q expansion. Namely, all patterns with k digits appear with the same frequency. Show that x is q-normal if and only if the sequence $\{q^n x\}$ is uniformly distributed mod 1.
Hint: To prove uniform distribution, verify the definition first for intervals of the form $[i/q^k, (i+1)/q^k)$ and then approximate a general interval by intervals with endpoints rational numbers, whose denominators are powers of q.

> First of all, we prove that if x is q-normal, then for every $k \in \mathbb{N}$ and $i \in \{0, \ldots, q^k - 1\}$ we have
>
> $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n \mathbb{1}_{[\frac{i}{q^k}, \frac{i+1}{q^k})}(x) = \frac{1}{q^k}. \tag{4}$$
>
> The left-hand side of (4) can be rewritten as
>
> $$\lim_{N\to\infty} \frac{\left|\left\{n \leq N \colon \{q^n x\} \in [\frac{i}{q^k}, \frac{i+1}{q^k}]\right\}\right|}{N} = \frac{1}{q^k}.$$

By the previous part, there exists a unique choice of digits $c_1, \ldots, c_k$ such that

$$\{q^n x\} \in [\frac{i}{q^k}, \frac{i+1}{q^k}] \iff a_{n+1}(x) = c_1, \ldots, a_{n+k}(x) = c_k.$$

Therefore, we can rewrite the last limit as

$$\lim_{N\to+\infty} \frac{|\{n \leq N : a_{n+1}(x) = c_1, \ldots, a_{n+k}(x) = c_k\}|}{N} = \frac{1}{q^k}$$

which holds by the definition of normality in base q. This proves our claim that (4) holds.
Now, we prove that if $A = [a, b)$ is a sub-interval of $[0, 1]$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n \mathbb{1}_A(x) = \mu(A). \tag{5}$$

Indeed, let $\mathcal{P}$ be the collection of Borel sets for which (5) holds. We know that $\mathcal{P}$ contains the family $\{\emptyset\} \cup \{[\frac{i}{q^k}, \frac{i+1}{q^k})\}_{k\in\mathbb{N}, i\in\{0,\ldots,q^k-1\}}$. In addition, if $A, B \in \mathcal{P}$ are disjoint, we have that $A \cup B \in \mathcal{P}$, since

$$\begin{aligned}\frac{1}{N} \sum_{n=1}^{N} T^n \mathbb{1}_{A\cup B}(x) = \frac{1}{N} \sum_{n=1}^{N} T^n (\mathbb{1}_A + \mathbb{1}_B(x)) = \frac{1}{N} \sum_{n=1}^{N} T^n \mathbb{1}_A(x) + \frac{1}{N} \sum_{n=1}^{N} T^n \mathbb{1}_B(x) \\ \to \mu(A) + \mu(B) = \mu(A \cup B).\end{aligned}$$

We conclude that $\mathcal{P}$ contains all intervals of the form $\left[\frac{i}{q^k}, \frac{j}{q^k}\right)$ or, equivalently, all half-open intervals with rational endpoints whose denominators are powers of q.
Let $A = [a, b) \subset [0, 1)$ be an arbitrary interval and let $\varepsilon > 0$. Then, we can find half-open intervals J_1, J_2 with endpoints that are rational numbers with denominators powers of q, such that $J_1 \subseteq A \subseteq J_2$ and such that $\mu(J_2) - \varepsilon < \mu(A) < \mu(J_1) + \varepsilon$. This follows from the fact that the rationals of the form m/q^k are dense in $[0, 1]$. Then, we have

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n \mathbb{1}_A(x) \geq \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n \mathbb{1}_{J_1}(x) > \mu(A) - \varepsilon$$

Using J_2 in place of J_1, we conclude that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n \mathbb{1}_A(x) < \mu(A) + \varepsilon.$$

Since ε was arbitrary, we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n \mathbb{1}_A(x) = \mu(A).$$

We can rewrite this as (recall $A = [a, b)$)

$$\lim_{N\to\infty} \frac{|\{n \leq N : \{q^n x\} \in [a, b)\}|}{N} \to \mu(A) = (b - a).$$

and, thus, $\{q^n x\}$ is uniformly distributed.

Remark: *Here, we note that* (5) *holds for intervals, but not for general measurable sets. In particular, for any sequence x_n, we can show that there exists a set of measure* 1 *that does not contain any of the elements x_n.*

We now prove the reverse implication. Assume $(q^n x)$ is uniformly distributed mod 1. Let $c_1, \dots, c_k \in \{0, \dots, q-1\}$ and let $i = q^{k-1}c_1 + \ldots + qc_{k-1} + c_k$. Using part (b), we conclude that

$$\frac{|\{n \leq N \colon a_{n+1} = c_1, \dots, a_{n+k} = c_k\}|}{N} = \frac{\left|\left\{n \leq N \colon \{q^n x\} \in [\frac{i}{q^k}, \frac{i+1}{q^k})\right\}\right|}{N}.$$

Since $q^n x$ is equidistributed, we have

$$\lim_{N \to +\infty} \frac{\left|\left\{n \leq N \colon \{q^n x\} \in [\frac{i}{q^k}, \frac{i+1}{q^k})\right\}\right|}{N} = \mu\left(\left[\frac{i}{q^k}, \frac{i+1}{q^k}\right)\right) = \frac{1}{q^k}.$$

The conclusion follows.